

# Space-time as multidimensional elastic plate

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## Abstract

It is suggested, that a curved 4-dimensional space-time manifold is a strained elastic plate in multidimensional embedding space-time. Its thicknesses along extradimensions are much less than 4-dimensional sizes. Reduced 4-dimensional free energy density of the strained plate in a weak strain case is similar to GR Lagrangian density of a gravitational field for the particular value of the Poisson coefficient of the plate. Dynamical equations of the theory are obtained by variation of the multidimensional free energy over displacement vector components  $\xi^A$ . In general case they are inhomogeneous bewave equations.

## 1 Introduction

In this article we suggest an alternative (to GR) approach to the description of space-time dynamics. It is based on the multidimensional formulation of elasticity theory (MET), which is a direct generalization of the standard 3-dimensional elasticity theory. Then the space-time continuum should be treated as multidimensional physical medium with elastic properties. Dimension of the observable space-time leads us to the concept of *4-dimensional plate*. It is a multidimensional body, whose sizes along 4 dimensions are much greater then in other ones. We suppose, that these extradimensions have certain "thicknesses" (negligibly small) and consider curvature of space-time as a manifestation of mechanical straining of this multidimensional plate. In some sense similar philosophy have been used in [1]-[5].

To compare our results with GR, we use embedding theory formalism [6]-[10]. In section 2 we present some usefull facts from the embedding theory. The expression for the gravitational field action in terms of displacement vector components derivatives is obtained.

Section 3 is devoted to the multidimensional generalization of the standard elasticity theory and to the theory of a thin multidimensional plate straining. The formula for the elastic free energy of the multidimensional plate, integrated over extra coordinates, is derived. Then we compare variational functionals of GR and MET.

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In section 4 dimensional analysis is made and some numerical evaluations for Young's modulus of the multidimensional plate are obtained.

In section 5 we variate free energy functional over multidimensional displacement vector components. Multidimensional generalization of Sophy-Zhermen equation and boundary conditions are obtained.

In section 6 Schwarzschild solution is analyzed from the viewpoint of MET.

## 2 Embedding theory in deformation representation

Let us consider pseudoeuclidean space of  $N + 4$  dimensions, where  $p + 1$  ones are time-like and  $q + 3$  — space-like:  $M_{p+1,q+3}$  ( $p + q = N$ ). Its metric can be defined as  $\eta_{AB} = \epsilon_A \delta_{AB}$ ,  $A, B = \overline{1, N + 4}$ ,  $\epsilon_A = \pm 1$ . Hereafter big Latin letters  $A, B, C, \dots$  denote tensor components with respect to the group of rotation  $O(p + 1, q + 3)$  in  $M_{p+1,q+3}$ , Greek indices  $\alpha, \nu, \lambda, \dots = \overline{0, 1, 2, 3}$  — are tensor components with respect to a group of general coordinate transformations in  $M_{1,3}$  (and  $V_{1,3}$ ) and small Latin indices  $m, n, \dots = \overline{1, N}$  — are tensor components in subspace of  $M_{p+1,q+3}$ , orthogonal to  $M_{1,3}$ , where 4-dimensional plane  $M_{1,3}$ , contains one time-like and three space-like directions (fig.1). Using the transformations from the isometry group of  $M_{p+1,q+3}$

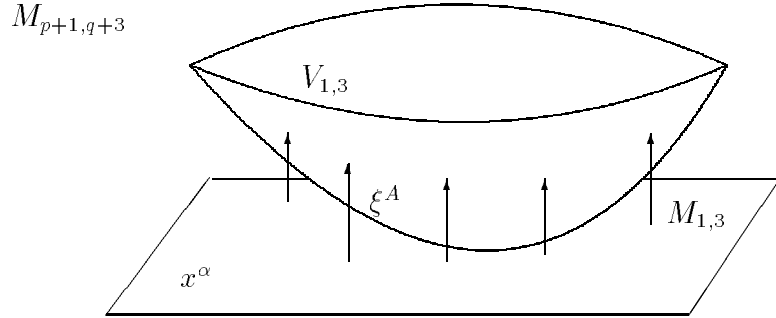


Figure 1: Embedding of  $M_{1,3}$  and  $V_{1,3}$  in  $M_{p+1,q+3}$  in deformation representation

and renumerating Cartesian coordinates in  $M_{p+1,q+3}$  we'll get the equation of this plane in the form:

$$x^m = 0, m = \overline{1, N}. \quad (1)$$

Then four Cartesian coordinates  $x^\alpha$  in  $M_{p+1,q+3}$  become Cartesian coordinates in  $M_{1,3}$ , and induced metric in  $M_{1,3}$  will be  $\eta_{\alpha\nu} = \text{diag}\{+1, -1, -1, -1\}$ .

Let  $\xi^A(x^\alpha)$  be smooth vector field defined on  $M_{1,3}$ , orthogonal to  $M_{1,3}$ . This vector field determines a deformation of  $M_{1,3}$  into some curved Riemannian manifold  $V_{1,3}$  (up to the parallel transition and rigid rotation in  $M_{p+1,q+3}$ ). The only nonzero components of  $\xi^A$  are  $\xi^m$  (due to (1) and orthogonality to  $M_{1,3}$ ). Metric on  $V_4$ , can be expressed in terms of displacement vector derivatives over coordinates  $x^\alpha$ :

$$ds^2 = \eta_{AB} dx^A dx^B = \eta_{\alpha\nu} dx^\alpha dx^\nu + \eta_{mn} \xi_{,\alpha}^m \xi_{,\nu}^n dx^\alpha dx^\nu = \eta_{AB} Y_\alpha^A Y_\nu^B dx^\alpha dx^\nu = g_{\alpha\nu} dx^\alpha dx^\nu, \quad (2)$$

where  $Y_\alpha^A = \delta_\alpha^A + \xi_{,\alpha}^A$  — four tangent to the  $V_{1,3}$  vector fields.

Let's introduce  $N$  unit vectors  $n_{m|}^A$  orthogonal to each other and to  $V_{1,3}$ :

$$\eta_{AB} n_{m|}^A n_{n|}^B = \epsilon_m \delta_{mn}; \quad (3)$$

$$\eta_{AB} n_{m|}^A Y_\alpha^B = 0, \quad (4)$$

where vertical bar separates the number of the vector from the coordinate index. Hereafter we use notations of [11]. Then at every point of  $V_{1,3}$  tangent basis  $\vec{Y}$  can be supplemented to a basis in  $M_{p+1,q+3}$ . The Cristoffel symbols in  $V_{1,3}$  are the following:

$$\Gamma_{\nu\lambda}^\alpha = Y^{A\alpha} Y_{A\nu,\lambda}. \quad (5)$$

Raising and lowering of Greek indices is made with  $g^{\alpha\nu}$  and  $g_{\alpha\nu}$ , Latin indices — with  $\eta_{AB}$ .

The covariant derivative of the tangent vector has the form:

$$Y_{\alpha;\nu}^A = Y_{\alpha,\nu}^A - \Gamma_{\alpha\nu}^\lambda Y_\lambda^A = Y_{\alpha,\nu}^A - Y_\lambda^B Y_{B\alpha,\nu} Y^{A\lambda}. \quad (6)$$

This derivative is the tensor on  $V_{1,3}$ , and is the vector in  $M_{p+1,q+3}$ , orthogonal to  $V_{1,3}$ . Due to this fact we can expand it with respect to  $n_{m|}^A$ :

$$Y_{\alpha;\nu}^A = \sum_m \epsilon_m \Omega_{m|\alpha\nu} n_{m|}^A. \quad (7)$$

The symmetric tensor  $\Omega_{m|\alpha\nu}$  is multidimensional generalization of a second fundamental form of a 2-dimensional surface in 3-dimensional space. Scalar product of (7) and  $n_{s|A}$  gives (taking into account (3) and (6)):

$$\Omega_{s|\alpha\nu} = n_{s|A} Y_{\alpha;\nu}^A = n_{s|A} Y_{\alpha,\nu}^A = n_{s|m} \xi_{\alpha,\nu}^m. \quad (8)$$

Integrability condition for Eq. (7) has the form:

$$2Y_{\alpha[\nu;\lambda]}^A = R_{\alpha\nu\lambda}^\sigma Y_\sigma^A. \quad (9)$$

We need only those conditions from (9), which give relation between curvature tensor of inner geometry and the quadratic combinations of  $\Omega_{m|\alpha\nu}$ :

$$R_{\alpha\nu\lambda\sigma} = \sum_{m=1}^N \epsilon_m (\Omega_{m|\alpha\lambda} \Omega_{m|\nu\sigma} - \Omega_{m|\alpha\sigma} \Omega_{m|\nu\lambda}), \quad (10)$$

which are called *Gauss equations*. From (10) and (8) one gets

$$R_{\alpha\nu\lambda\sigma} = \sum_{m=1}^N (\xi_{\alpha,\lambda}^n \xi_{\nu,\sigma}^l - \xi_{\alpha,\sigma}^n \xi_{\nu,\lambda}^l) \epsilon_m n_{m|n} n_{m|l} = (\xi_{\alpha,\lambda}^n \xi_{\nu,\sigma}^l - \xi_{\alpha,\sigma}^n \xi_{\nu,\lambda}^l) H_{nl}, \quad (11)$$

where  $H_{nl} \stackrel{\text{def}}{=} \sum_{m=1}^N \epsilon_m n_{m|n} n_{m|l}$  — is the projector on orthogonal to  $V_{1,3}$  directions. From (11) we can easily find Ricci tensor and curvature scalar.

Thus gravitational field action in GR can be written in terms of embedding variables:

$$S_g = -\frac{c^3}{16\pi G} \int R \sqrt{-g} d^4x = -\frac{c^3}{16\pi G} \int \{ (\xi_{\alpha,\lambda}^n \xi_{\nu,\sigma}^l - \xi_{\alpha,\sigma}^n \xi_{\nu,\lambda}^l) H_{nl} \} \sqrt{-g} d^4x. \quad (12)$$

It is clear now that embedding theory allows us to rewrite gravitational action in terms of derivatives of multidimensional displacement vector.

### 3 Multidimensional elasticity theory and strain of a multidimensional plate.

Let us consider some multidimensional body in  $M_{p+1,q+3}$  in unstrained state and a Cartesian coordinates system rigidly tied with it. Every point of the strained body displaces on some vector  $\vec{\xi}$ , which is a function of coordinates. The square of the length between two neighbour points of the strained body is given by the following expression

$$dl'^2 = dl^2 + 2\xi_{AB}dx^A dx^B \quad (13)$$

where symmetric tensor

$$\xi_{AB} = \frac{1}{2} \left( \frac{\partial \xi_A}{\partial x^B} + \frac{\partial \xi_B}{\partial x^A} + \frac{\partial \xi_M}{\partial x^A} \frac{\partial \xi^M}{\partial x^B} \right), \quad (14)$$

is called the *strain tensor* (see for example [12]). Quadratic terms in (14) are negligibly small for small strains and can be omitted.

Inner forces, induced by the deformation, act on a surface of the separated volume only and, consequently, the resulting force, acting on the chosen volume, has the form:

$$F_A = \int_V f_A dV = \int_V \frac{\partial \sigma_{AB}}{\partial x^B} dV = \oint_{\partial V} \sigma_{AB} ds^B, \quad (15)$$

where  $f_A$  —  $A$ -th component of the force volume density, the  $\sigma_{AB}$  — *stress tensor* and  $f_A = \sigma_{AB}^B$ ,  $\partial V$  — is the hypersurface, bounding separated volume. Tensor  $\sigma_{AB}$  is not uniquely determined from (15) and it can always be written in the symmetric form.

Equilibrium equations in the absence of an external forces are:

$$\frac{\partial \sigma_{AB}}{\partial x^B} = 0. \quad (16)$$

The boundary conditions of equilibrium equations in elasticity theory has the form:

$$\sigma_{AB} n^B = P_A, \quad (17)$$

where  $P_A$  — hypersurface force density,  $\vec{n}$  — unit vector normal to the surface.

From the multidimensional generalization of the first law of thermodynamics

$$dF = -SdT + \sigma_{AB} d\xi^{AB}$$

one can find:

$$\sigma_{AB} = \left( \frac{\partial F}{\partial \xi^{AB}} \right)_T \quad (18)$$

where  $\delta A = -\sigma_{AB} \delta \xi^{AB}$  is the work of inner elastic forces,  $F = E - TS$  — multidimensional free energy density,  $E$  — multidimensional internal energy density,  $T$  — multidimensional absolute temperature,  $S$  — multidimensional entropy density.

Equations of static elasticity theory can be obtained by variation of  $F$  over displacement vector components. For the isothermic weak strain case free energy can be expressed in terms of strain tensor components:

$$F = \frac{\lambda}{2}(\xi_A^A)^2 + \mu\xi_{AB}\xi^{AB} = \mu\left(\xi_{AB} - \frac{1}{n}\eta_{AB}\xi_C^C\right)^2 + \frac{K}{2}(\xi_C^C)^2 = \frac{E}{2(1+\sigma)}\left(\xi_{AB}\xi^{AB} + \frac{\sigma}{1-(n-1)\sigma}(\xi_C^C)^2\right), \quad (19)$$

where  $\lambda$  and  $\mu$  are called *Lame coefficients*,  
 $K = \lambda + (2/n)\mu$  — *stretch modulus*,  $\mu$  — *shear modulus*,

$$E = \frac{n^2 K \mu}{(1/2)n(n-1)K + \mu}, \quad \sigma = \frac{1}{2} \frac{(nK - 2\mu)}{[(1/2)n(n-1)K + \mu]}$$

— *Young's modulus and Poisson coefficient* respectively. Here  $n$  — is the total number of dimensions —  $N + 4$ . To clarify physical meaning of Young's modulus and Poisson coefficient let us consider a simple stretch of a bar along his axe of symmetry (axe  $x^1$ ). Then  $\xi_{11} = p/E$ ,  $\xi_{AB} = -\epsilon_1 \eta_{AB} \sigma \xi_{11}$ ,  $A, B \neq 1$ , where  $p$  — stretch force, acting on the unit hyphersurface of the bar end.

Taking (18) and (19) one can obtain expression for a stress tensor in linear elasticity theory:

$$\sigma_{AB} = 2\mu\xi_{AB} + \lambda\eta_{AB}\xi_C^C = \frac{E}{1+\sigma}\left(\xi_{AB} + \frac{\sigma}{1-(n-1)\sigma}\eta_{AB}\xi_C^C\right). \quad (20)$$

Let's consider a weak strain of a thin 4-dimensional plate. Let  $h_m$  be thickness of the plate in the  $m$ -th extradimension orthogonal to  $M_{1,3}$ . Suppose, that the middle (in thickness) plane of the unstrained plate coincides with  $M_{1,3}$ -plane and  $V_{1,3}$  is the surface of weakly strained plate. The middle surface in the case of weak straines is usually called *newtral surface* because tangent stresses on it are equal to zero. Putting the origin of  $(p+1, q+3)$ -dimensional Cartesian coordinate system on the newtral surface (Fig.2) one can treat  $\xi^A$  vector field as the displacement vector of newtral surface points in the orthogonal to  $M_{1,3}$  directions. As in standard 3-dimensional elasticity theory we can put  $P_A$  in (17) to zero, (because  $P_A$  is negligibly small in comparison with  $\sigma_{AB}$ ) and consider  $n_m^A$  as orthogonal vectors to  $M_{1,3}$ -plane (because of smallness of plate bending). Thus instead of (17) we have

$$\sigma_{nA} = 0. \quad (21)$$

From (20) we get the following system of equations for the components of weak strain tensor  $\xi_{AB} = (1/2)(\xi_{A,B} + \xi_{B,A})$

$$\xi_{mn} = \xi_{\alpha n} = 0, \quad (m \neq n); \quad 2\mu\xi_{mm} + \lambda\xi_A^A\epsilon_m = 0.$$

Its solution is

$$\xi_\alpha = -x^m \xi_{m,\alpha}$$

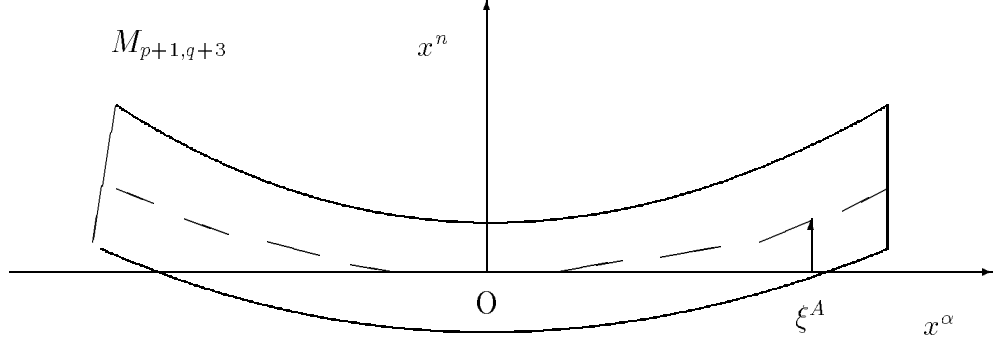


Figure 2: Straining of a thin plate

Then for  $\xi_{AB}$  we obtain ( $\mu \neq 0$ ,  $N\lambda + 2\mu \neq 0$ ):

$$\xi_{\alpha n} = \xi_{mn} = 0; \quad \xi_{\alpha\beta} = -\xi_{m,\alpha,\beta} x^m; \quad (22)$$

$$\xi_{mm} = -\epsilon_m \frac{\lambda S}{N\lambda + 2\mu}; \quad S = \xi_{\alpha}^{\alpha} = -x^m \xi_{m,\alpha}^{\alpha}.$$

Substituting (22) into (19) we get

$$F = \mu x^m x^n \{ \xi_{m,\alpha,\nu} \xi_{n,\alpha}^{\alpha,\nu} + f \xi_{m,\alpha}^{\alpha} \xi_{n,\nu}^{\nu} \}, \quad (23)$$

where  $f = \lambda / (N\lambda + 2\mu)$ .

To obtain expression for total free energy of the plate one should integrate (23) over its  $N + 4$ -dimensional volume:

$$F_{\text{pl}} = \frac{\mu H_N h_m^2}{12} \eta^{mn} \int \sqrt{-g} d^4 x \{ \xi_{m,\alpha,\nu} \xi_{n,\alpha}^{\alpha,\nu} + f \xi_{m,\alpha}^{\alpha} \xi_{n,\nu}^{\nu} \}. \quad (24)$$

where

$$\delta^{mn} H_N \frac{h_m^2}{12} = \delta^{mn} \left( \prod_{n=1}^N h_n \right) \frac{h_m^2}{12} = \prod_{p=1}^N \int_{-h_p/2}^{h_p/2} dx^p x^m x^n,$$

$H_N$  is the product of all thicknesses of the plate,  $\delta^{mn}$  — Kronecker symbol,  $\eta^{mn}$  — are orthogonal to  $M_{1,3}$  components of  $\eta_{AB}$ .

Comparing (24) and (12) we may note their remarkable similarity. Let's find the conditions under which (24) and (12) will be identical:

1. The coefficient  $f$  in (24) must be equal to  $-1$ :

$$f = -1. \quad (25)$$

Taking into account expression for generalized multidimensional Lamé coefficients

$$K = \frac{E}{(1 - (n-1)\sigma)n}; \quad \mu = \frac{E}{2(1+\sigma)}; \quad \lambda = \frac{E\sigma}{(1+\sigma)(1 - (n-1)\sigma)}; \quad (26)$$

we get the first condition

$$\sigma = \frac{1}{n - N - 2} \quad (27)$$

and for  $n = N + 4$  one can easily obtain  $\sigma = 1/2$ .

2. In the case of weak deformation we can put in (12):

$$H_{mn} \approx \eta_{mn}$$

It yields the second conditions

$$h_m = h, \quad (28)$$

i.e. plate must have equal thicknesses in all extradimensions. Easy to see, that last condition is connected with norming conditions (3), and simple renormalization of vector  $n_{m|}^A$ :

$$\eta_{AB} n_m^A n_n^B = h_m^2 \varepsilon_m \delta_{mn}$$

gives us an opportunity to formulate the theory without the condition (28) (we use (3) to save the standard form of equations of embedding theory).

This comparison leads us directly to the idea, that *our observable macroscopic 4-dimensional space-time can be described as the thin multidimensional plate, "made" of some specific material. Its thicknesses are much less, then its 4-dimensional sizes and its free energy functional is defined by (24).*

## 4 Dimensional analysis.

Equating (12) and (24) and introducing constant  $A$  we get (with the conditions (25) and (28)):

$$\frac{c^3}{16\pi G} \equiv A \frac{\mu h^{N+2}}{12} \quad (29)$$

where  $A$  is measured in seconds. There are only two combinations of fundamental constants  $G, \hbar, c, e, h$ , ( $h$  — thickness of the plate), which have appropriate dimension "second":

$$\tau = h/c \text{ and } \tau_{pl} = \sqrt{G\hbar/c^5} \approx 10^{-44} s,$$

where  $\tau_{pl}$  is Planck time. From  $\pi$ -theorem in dimension theory (see for example [13])

$$A = \varphi_1\left(\frac{e^2}{\hbar c}, \frac{\tau}{\tau_{pl}}\right)\tau + \varphi_2\left(\frac{e^2}{\hbar c}, \frac{\tau}{\tau_{pl}}\right)\tau_{pl}, \quad (30)$$

where  $\varphi_1, \varphi_2$  — two arbitrary functions of two dimensionless combination of fundamental constants. In the simplest case of multiplicative dependence of  $A$  on the fundamental constants we have

$$E h^{N+S+3} \sim c^{4-(3/2)S-P} G^{S/2-1} \hbar^{S/2-P} e^{2P}, \quad (31)$$

Here  $S, P$  — arbitrary real numbers. From (31) one can get a wide class of relations between elastic constants and fundamental ones. Let us consider two most interesting cases:

a) Assuming, that extradimensions have sizes of the Planck length  $h_{pl} = \sqrt{G\hbar/c^3} \approx 10^{-33}\text{cm}$  (as in Kaluza-Klein theories), we obtain

$$E \sim c^{3/2N-P+17/2} G^{-5/2-N/2} \hbar^{-P-N/2-3/2} e^{2P} \sim 10^{144+35N-3P} \text{Pa} \quad (32)$$

We see, that space-time plate in this case possess by a huge stiffness (compare with chromium  $\sim 10^{12}\text{Pa}$ ). It gives natural explanation of observable "flatness" of space-time in local regions.

b) For different values of parametres  $P$  and  $S$  one gets:

1) If  $P = 0$ ,  $S = 0$  then constants  $e$  and  $\hbar$  vanish

$$Eh^{N+3} \sim \frac{c^4}{G}.$$

This case is the most realistic (due to the absence of  $e$  and  $\hbar$  in the classical approach to gravity) and connected with old Sacharov's idea [2] about possible elasticity of space-time.

2) If  $S = 2$ ,  $P = 1$  then constants  $G, \hbar, c$  vanish

$$Eh^{N+5} \sim e^2;$$

3) If  $S = -N - 3$ ,  $P = -(N + 3)/2$  then  $e$  and  $\hbar$  vanish

$$E \sim \frac{ec^2}{\sqrt{G}} \left/ \left( \frac{\sqrt{G}e}{c^2} \right)^{N+4} \right. \sim 10^{151+36N} \text{Pa};$$

4) If  $P = 0$ ,  $S = 2$  then  $e$  and  $G$  vanish

$$Eh^{N+5} \sim c\hbar;$$

5) If  $P = 0$ ,  $S = 8/3$  then  $e$  and  $\hbar$  vanish

$$Eh^{N+17/3} \sim G^{1/3} \hbar^{4/3};$$

6) If  $P = 0$ ,  $S = -N - 3$  then  $e$  and  $\hbar$  vanish

$$E \sim \sqrt{\frac{\hbar c^5}{G}} \left/ \left( \frac{G\hbar}{c^3} \right)^{(N+4)/2} \right. \sim 10^{144+34N} \text{Pa};$$

7) Constants  $G$  and  $\hbar$  cannot be excluded from (31) simultaneously;

8) If  $S = -N - 3$ ,  $P = 4 + (3/2)(N + 3)$  then  $c$  and  $\hbar$  vanish

$$E \sim \left( \frac{e^5}{\sqrt{G}\hbar^2} \right) / \left( \frac{\sqrt{G}\hbar^2}{e^3} \right)^{N+4} \sim 10^{118+30N} \text{Pa}.$$

Presented analysis allows us to formulate the second hypothesis, which develops the first one: *Space-time plate is a very stiff body in multidimensional space-time. The Newtonian gravitational constant and elastic constants of the plate are related by algebraic equation (in the most realistic case:  $Eh^{N+3} \sim 1/\alpha$ .)*



## 5 Equilibrium equation and boundary conditions

Assume (as in standard 3-dimensional approach), that the volume of the plate does not change when bending. Then  $\sqrt{-g} \approx 1$ . Varying functional  $F$  over  $\xi^m$  we have:

$$\begin{aligned} \delta F &= \frac{\mu H_N h_m^2}{12} \eta_{mn} \delta \int d^4 x \{ \xi_{,\alpha,\nu}^m \xi_{,\nu,\alpha}^{n\alpha} + f \square \xi^m \square \xi^n \} = \\ &= \frac{\mu H_N h_m^2}{12} \eta_{mn} 2\delta \int d^4 x \left\{ \frac{f+1}{2} \square \xi^m \square \xi^n + \sum_{\alpha \neq \nu, \alpha < \nu} (\xi_{,\alpha,\nu}^m \xi_{,\nu,\alpha}^{n\alpha} - \xi_{,\alpha,\alpha}^m \xi_{,\nu,\nu}^{n\alpha}) \right\} = \\ &= D_m \eta_{mn} \delta \int d^4 x \left\{ \frac{1}{2} \square \xi^m \square \xi^n + \frac{1}{f+1} \sum_{\alpha \neq \nu, \alpha < \nu} (\xi_{,\alpha,\nu}^m \xi_{,\nu,\alpha}^{n\alpha} - \xi_{,\alpha,\alpha}^m \xi_{,\nu,\nu}^{n\alpha}) \right\}, \quad (33) \end{aligned}$$

where  $\square \equiv \partial^\alpha \partial_\alpha$  — wave operator,

$$D_m = \frac{\mu H_N h_m^2 (f+1)}{6} = \frac{E H_N h_m^2}{12(1+\sigma)} \cdot \frac{1+\sigma(N-n+2)}{1+\sigma(N-n+1)}$$

is the cylindrical stiffness factor of the plate in  $m$ -th extradimension<sup>1</sup>. Variation of the first term in (33) gives

$$\delta \frac{1}{2} \int d^4 x (\square \xi^m)^2 = \int_\Sigma d^4 x \square^2 \xi^m \delta \xi^m + \oint_{\partial \Sigma} d^3 S^\alpha \square \xi^m \delta \xi_{,\alpha}^m - \oint_{\partial \Sigma} d^3 S^\alpha \square \xi_{,\alpha}^m \delta \xi^m, \quad (34)$$

where  $\Sigma$  is 4-dimensional plate surface,  $\partial \Sigma$  is 3-dimensional bound of  $\Sigma$ . Variation of the second term in (33) yields:

$$\delta \int d^4 x \sum_{\alpha \neq \nu, \alpha < \nu} (\xi_{,\alpha,\nu}^m \xi_{,\nu,\alpha}^{m\alpha} - \xi_{,\alpha,\alpha}^m \xi_{,\nu,\nu}^{m\alpha}) = \oint_{\partial \Sigma} d^3 S^\alpha w_\alpha^m, \quad (35)$$

where  $w_\alpha^m = \delta \xi_{,\alpha,\lambda}^m \xi_{,\lambda,\alpha}^{m\alpha} - \delta \xi_{,\alpha,\alpha}^m \xi_{,\lambda,\lambda}^{m\alpha}$ . Variation of potential energy of the plate (due to applied bending forces) is  $\delta U = - \int d^4 x P_n \delta \xi^n$ . Finally we obtain:

$$\begin{aligned} \delta F_{pl} + \delta U &= \int_\Sigma d^4 x (D_m \square^2 \xi_m - P_m) \delta \xi^m - D_m \oint_{\partial \Sigma} d^3 S^\alpha \square \xi_{m,\alpha} \delta \xi^m + \\ &+ \frac{D_m}{1+f} \oint_{\partial \Sigma} d^3 S^\alpha (f \eta_{\alpha\lambda} \square \xi_m + \xi_{m,\alpha,\lambda}) \delta \xi^{m\lambda} \end{aligned} \quad (36)$$

Equating (36) to zero we get equilibrium equations

$$D_m \square^2 \xi^m = P^m, \quad (37)$$

where there is no summation over  $m$ . Eqs. (37) are multidimensional generalization of 3-D Sophy-Zhermen equilibrium equation. Integrals over  $\partial \Sigma$  give the boundary conditions for displacement vector and its derivatives. If all  $D_m = 0$ , then we get GR theory. In terms of  $\xi^m$  functional  $S_g$  (in linear approximation) is a full divergence. Thus only boundary integrals do not vanish in (36). It means, that there is no restrictions on (small) displacement within  $\partial \Sigma$ : any small bending of the plate is admissible if it meets boundary conditions. The similar situation for ordinary 2-D plate in standard 3-D elasticity theory can be obtained in unphysical limit:  $\sigma \rightarrow \infty$ .

<sup>1</sup>For  $n = 3$   $N = 1$  we have standard formulae ([12]).

## 6 Strained plate for Schwarzschild metric

First let us prove equivalence of deformation representation of embedding theory and standard representation of [11]. Let the set of functions  $\varphi^A(\zeta^\alpha)$ , defining isometric embedding of  $g_{\alpha\nu}$  in  $M_{p+1,q+3}$ , is given, i.e:

$$ds^2 = \eta_{AB} dx^A dx^B = \eta_{AB} \varphi_{,\alpha}^A \varphi_{,\nu}^B d\zeta^\alpha d\zeta^\nu = g_{\alpha\nu} d\zeta^\alpha d\zeta^\nu. \quad (38)$$

where  $\zeta^\alpha$  — arbitrary 4-dimensional coordinates. Let four functions  $\varphi^\alpha$  (from the set  $\varphi^A$ ) be equal to new 4-D coordinates:

$$\varphi^\alpha \equiv x^\alpha. \quad (39)$$

Here (without loosing of generality)  $\varphi^0$  corresponds to time-like dimension of  $M_{p+1,q+3}$  and  $\varphi^1, \varphi^2, \varphi^3$  - to space-like ones. Besides we assume, that Jacobian  $D\varphi/D\zeta \neq 0$ . Thus one can solve eq. (39) with respect to  $\zeta$ :

$$\zeta^\alpha = \zeta^\alpha(x). \quad (40)$$

Substituting (40) into the rest functions  $\varphi^m$  we get:

$$\varphi^m = \varphi^m(\zeta(x)) = \bar{\varphi}^m(x). \quad (41)$$

Then metric (38) takes the form:

$$ds^2 = \eta_{\alpha\nu} dx^\alpha dx^\nu + \eta_{mn} \bar{\varphi}_{,\alpha}^m \bar{\varphi}_{,\nu}^n dx^\alpha dx^\nu. \quad (42)$$

Comparing (42) and (2) we find that (42) is written in deformation representation with functions  $\bar{\varphi}^m(x)$  as components of displacement vector.

It is clear, that this representation is not unique, because of: 1) uncertainty of an isometric embedding and 2) freedom in choosing of functions  $\varphi^\alpha$  as 4-dimensional coordinates.

It is well known [14], that Schwarzschild metric:

$$ds^2 = \left(1 - \frac{r_g}{r}\right) dt^2 - \frac{dr^2}{1 - \frac{r_g}{r}} - r^2 d\Omega^2$$

admits isometric embedding of class 2 in  $M_{2,4}$  with the following embedding functions:

$$X^1 = \sqrt{1 - \frac{r_g}{r}} \cos t; \quad X^2 = \sqrt{1 - \frac{r_g}{r}} \sin t; \quad X^3 = f(r); \quad (43)$$

$$X^4 = r \sin \theta \cos \varphi; \quad X^5 = r \sin \theta \sin \varphi; \quad X^6 = r \cos \theta,$$

where  $f(r)$  may be found from the equation:  $f'^2 = (r_g^2/4r^4 + r_g/r) / (1 - r_g/r)$ . Metric in  $M_{2,4}$  has the form:  $\eta_{AB} = \text{diag}(+, +, -, -, -, -)$ .

Let  $(X^1, X^4, X^5, X^6)$  be coordinates on  $M_{1,3}$ , then  $X^2 \equiv \xi^1$ ,  $X^3 \equiv \xi^2$  take the form:

$$\xi^1 = \sqrt{1 - \frac{r_g}{r} - t^2}; \quad \xi^2 = f(r). \quad (44)$$

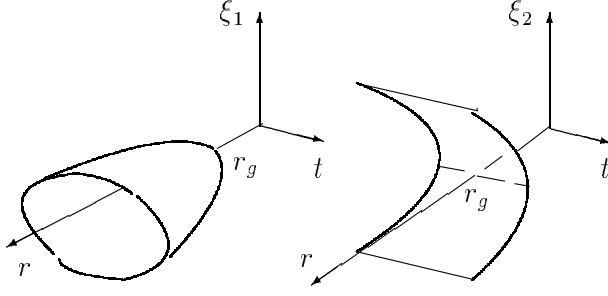


Figure 3: Strained 4-dimensional plate for Schwarzschild solution

The corresponding strained plate is shown on figures 3. The first figure shows the form of the strained plate in  $\xi^1$ -dimension: it is a paraboloid of revolution with respect to  $r$ -axis. Second figure shows the form of the plate in  $\xi^2$ -dimension: it is an infinite cylindrical surface. Every point of this surfaces is a unit sphere in 3-dimensional space section. There is no 4-dimensional space-time for  $r < r_g$ , so such 4-dimensional coordinate system is incomplit.

Since the Schwarzschild metric is the solution of Einstein equation then variational principle (36) gives boundary conditions only:

$$\frac{EH_N h_m^2}{12(1 + \sigma)} \oint_{\partial \Sigma} d^3 S^\alpha (\Box \xi_m \delta_\alpha^\lambda - \xi_{m,\alpha}{}^\lambda) \delta \xi_{,\lambda}^m = 0, \quad (45)$$

For  $n = 6$  the coefficient in (45) is nonzero, then the integral equals zero. If plate has free or simply supported ends, then variations of displacement vector and its derivatives are independent and

$$\Box \xi_m \delta_\alpha^\lambda - \xi_{m,\alpha}{}^\lambda = 0 \quad (46)$$

Direct calculation shows, that (46) is false and we can conclude, that if  $n = 6$ , then the bending plate (corresponding to Schwarzschild solution) must have pinned ends at infinity (as in GR).

It is clear from Fig.3, that the plate, corresponding to Swarcshild metric, is bent considerably, so the linear approximation of MET in this case is not justified. Though in the case of a weak gravitational fields (far from sources) it is absolutely correct. The detailed comparison of MET and GR will be continued in the next paper.

## 7 Conclusion

The proposed approach of MET gives us natural generalization of GR in the case of weak gravity. The multidimensional elasticity theory of strong deformation must be developed to describe strong gravitational fields.

In general case of weak deformation we get different from GR equations for space-time dynamics. The strong deformation theory may give natural physical arguments for including high curvature terms into Lagrangian of gravitational field.

Multidimensional generalization of classical mechanics and thermodynamics (multidimensional force, pressure, mass, temperature, enthropy) would be discussed separately (see for example [15]-[17]).

## Aknowlegments

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